The Exact Solutions of Some Multidimensional Generalizations of the Fokker-Planck Equation used by R.Friedrich and J.Peinke for the Description of a Turbulent Cascade

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Abstract

Some multidimensional generalizations of the Fokker-Planck equation used by R. Friedrich and J. Peinke for the description of a turbulent cascade as a stochastic process of Markovian type, are considered. The exact solutions of the Cauchy problems for these equations are found with the operator methods.

1 Introduction

The understanding of the turbulence is one of the main unsolved problems of classical physics, in spite of the more than 250 years of strong investigations initiated by D.Bernoulli and L.Euler.

In the stochastic approach to turbulence [1], [2] the turbulent cascade is considered as a stochastic process, described by the probability distribution $P(\lambda, v)$, where λ and v are the appropriate scaled length and the velocity increment respectively. Recently [3] R.Friedrich and J.Peinke presented experimental evidence that the probability density function $P(\lambda, v)$ obeys a Fokker-Planck equation (FPE) [4] (see fig.1 and fig.2 in [3]):

$$\frac{\partial P(\lambda, v)}{\partial \lambda} = \left[-\frac{\partial}{\partial v} D^1(\lambda, v) + \frac{\partial^2}{\partial v^2} D^2(\lambda, v) \right] P(\lambda, v), \tag{1}$$

where the drift and duffusion coefficients D^1 and D^2 respectively are derived by analysis of experimental data of a fluid dynamical experiment (see fig.3 in [3]).

In their paper Friedrich and Peinke use the following approximations for the drift and diffusion terms respectively

$$D^1 = -a\, v, \qquad a > 0; \qquad D^2 = c\, v^2, \qquad c > 0\,,$$

where a and c do not depend on λ.

In our previous paper [5], using the method of M.Suzuki [6] and the Feynman's disentangling techniques [7] we gave the exact solution of the Cauchy problem for the Eq. (1) with more realistic approximations (see fig.3 in [3]) for D^1 and D^2 :

$$D^1 = -a(\lambda)\, v, \qquad a(\lambda) > 0; \qquad D^2 = c(\lambda)\, v^2, \qquad c(\lambda) > 0. \tag{2}$$

In Section 2 of this paper we consider the Cauchy problem for the following n-dimensional generalization of the Eqs. (1), (2):

$$\frac{\partial P}{\partial \lambda} = b_0(\lambda)P(\lambda,\mathbf{v}) + b_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}P(\lambda,\mathbf{v}) + c(\lambda)\mathbf{v}^2\Delta_{\mathbf{v}}P(\lambda,\mathbf{v}), \tag{3}$$

$$P(0,\mathbf{v}) = \varphi(\mathbf{v}).$$

This equation is derived from the n-dimensional FPE

$$\frac{\partial P}{\partial \lambda} = -\nabla_{\mathbf{v}}\cdot\mathbf{D}^1(\lambda,\mathbf{v})P(\lambda,\mathbf{v}) + \sum_{i,j=1}^{n}\frac{\partial^2}{\partial v_i \partial v_j}D^2_{ij}(\lambda,\mathbf{v})P(\lambda,\mathbf{v}) \tag{4}$$

with the drift vector $\mathbf{D}^1(\lambda,\mathbf{v}) = -a(\lambda)\mathbf{v}$ and the diffusion tensor $\hat{D}^2(\lambda,\mathbf{v}) = c(\lambda)\mathbf{v}^2\hat{1}$. (Consequently $b_0(\lambda) = n[a(\lambda)+2c(\lambda)]$ and $b_1(\lambda) = a(\lambda)+4c(\lambda)$).

In Section 3 we find the exact solution of the Cauchy problem

$$\frac{\partial P}{\partial \lambda} = b_0(\lambda)P(\lambda,\mathbf{v}) + b_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}P(\lambda,\mathbf{v}) + c(\lambda)(\mathbf{v}\cdot\nabla_{\mathbf{v}})^2P(\lambda,\mathbf{v}), \tag{5}$$

$$P(0,\mathbf{v}) = \varphi(\mathbf{v}).$$

The Eq. (5) is derived from the FPE (4) with the drift vector $\mathbf{D}^1(\lambda,\mathbf{v}) = -a(\lambda)\mathbf{v}$ and the diffusion tensor $\hat{D}^2(\lambda,\mathbf{v}) = c(\lambda)\mathbf{v}\mathbf{v}$. (Consequently $b_0(\lambda) = na(\lambda) + (n^2+n)c(\lambda)$ and $b_1(\lambda) = a(\lambda) + (2n+1)c(\lambda)$).

Because of the analytic expressions of the diffusion tensor, one may regard the cases (3) and (5) respectively as the "isotropic" and "degenerate anisotropic" ($\det\hat{D} = 0$) ones.

2 Exact Solution of the Cauchy Problem (3)

In the spirit of the operational methods (see [8]–[11]) we have for the solution of the problem (3)

$$P(\lambda,\mathbf{v}) = \left(exp_+\int_0^\lambda \left[b_0(s) + b_1(s)\mathbf{v}\cdot\nabla_{\mathbf{v}} + c(s)\mathbf{v}^2\Delta_{\mathbf{v}}\right]\mathrm{d}s\right)\varphi(\mathbf{v}), \tag{6}$$

where the symbol $exp_+\int_0^\lambda \hat{C}(s)\mathrm{d}s$ designates the V.Volterra ordered exponential

$$exp_+\int_0^\lambda \hat{C}(s)\mathrm{d}s = \hat{1} + \lim_{n\to\infty}\sum_{k=1}^{n}\int_0^\lambda \mathrm{d}\lambda_1\int_0^{\lambda_1}\mathrm{d}\lambda_2\ldots\int_0^{\lambda_{k-1}}\mathrm{d}\lambda_k\hat{C}(\lambda_1)\hat{C}(\lambda_2)\ldots\hat{C}(\lambda_k). \tag{7}$$

The linearity of the integral and the explicit form of the operators in (6) permit to write the solution $P(\lambda, \mathbf{v})$ in terms of the usual, not ordered, operator valued exponent

$$P(\lambda, \mathbf{v}) = \mathrm{e}^{\beta_0(\lambda)}\, \mathrm{e}^{\beta_1(\lambda)\mathbf{v}.\nabla_{\mathbf{v}} + \gamma(\lambda)\mathbf{v}^2\Delta_{\mathbf{v}}}\varphi(\mathbf{v}), \tag{8}$$

where for convenience we have denoted

$$\beta_j(\lambda) = \int_0^\lambda b_j(s)\mathrm{d}s, \quad (j = 0, 1); \qquad \gamma(\lambda) = \int_0^\lambda c(s)\mathrm{d}s. \tag{9}$$

Consequently (from now on "$'$" means $\frac{\mathrm{d}}{\mathrm{d}t}$) :

$$\beta_j(0) = 0, \quad \beta_j'(\lambda) = b_j(\lambda), \quad (j = 0, 1); \qquad \gamma(0) = 0, \quad \gamma'(\lambda) = c(\lambda). \tag{10}$$

Since

$$\Big[\, \beta_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}\ ,\ \gamma(\lambda)\mathbf{v}^2\Delta_{\mathbf{v}} \,\Big] = 0 \tag{11}$$

the solution (8) can be written in the form:

$$P(\lambda, \mathbf{v}) = \mathrm{e}^{\beta_0(\lambda)}\mathrm{e}^{\beta_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}}\mathrm{e}^{\gamma(\lambda)\mathbf{v}^2\Delta_{\mathbf{v}}}\varphi(\mathbf{v}) = \mathrm{e}^{\beta_0(\lambda)}\mathrm{e}^{\gamma(\lambda)\mathbf{v}^2\Delta_{\mathbf{v}}}\mathrm{e}^{\beta_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}}\varphi(\mathbf{v}) \tag{12}$$

To write the expression (12) in a final form we will use the following formulae for acting of the pseudodifferential operators [12] - [14] $\mathrm{e}^{\beta\mathbf{v}\cdot\nabla_{\mathbf{v}}}$ and $\mathrm{e}^{\tau\mathbf{v}^2\Delta_{\mathbf{v}}}$ on arbitrary functions of $\mathbf{v}$:

$$\mathrm{e}^{\beta(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}} f(\mathbf{v}) = f\left(\mathbf{v}\mathrm{e}^{\beta(\lambda)}\right) \tag{13}$$

and (for $\tau(\lambda) > 0$)

$$\mathrm{e}^{\tau(\lambda)\mathbf{v}^2\Delta_{\mathbf{v}}} g(\mathbf{v}) = \frac{\mathrm{e}^{\tau(\lambda)\hat{\Lambda}_\theta}}{\sqrt{4\pi\tau(\lambda)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4\tau(\lambda)}} g\left(\mathbf{v}\mathrm{e}^{(n-2)\tau(\lambda)\pm s}\right) \mathrm{d}s. \tag{14}$$

Here $\hat{\Lambda}_\theta$ is the operator of Laplace-Beltrami on the sphere S_1 in $\mathcal{R}^n$ [15]–[17]:

$$\Delta_{\mathbf{v}} = \frac{\partial^2}{\partial v^2} + \frac{n-1}{v}\frac{\partial}{\partial v} + \frac{1}{v^2}\hat{\Lambda}_\theta \tag{15}$$

$$\hat{\Lambda}_\theta Y_{l,n}^{(k)} = \lambda_l Y_{l,n}^{(k)}, \quad \lambda_l = -l(l + n - 2), \quad l = 0, 1, 2, ..., \tag{16}$$

$$k = 1, 2, 3, ...d_{l,n}, \qquad d_{l,n} = \frac{(2l + n - 2)(n + l - 3)}{(n-2)!\ l!}.$$

The eigenfunctions $Y_{l,n}^{(k)}(\theta)$ are the spherical (harmonic) functions on S_1 which constitute the orthonormal basis in $L_2(S_1)$, i.e for any function g($\mathbf{v}$) we have:

$$g(\mathbf{v}) \equiv g(v, \theta) = \sum_{l=0}^{\infty}\sum_{k=1}^{d_{l,n}} g_{l,k}(v) Y_{l,n}^{(k)}(\theta)$$

$$g_{l,k}(v) = \left(Y^{(k)}_{l,n}, g\right) = \int\limits_{S_1} \overline{Y^{(k)}_{l,n}(\theta)} g(v,\theta)\mathrm{d}S_1, \tag{17}$$

where

$$\mathbf{v} = (|v|, \theta_1, \theta_2 \cdots \theta_{n-1}) = (v, \theta), \quad \theta_1, \cdots \theta_{n-2} \in [0,\pi], \quad \theta_{n-1} \in [0, 2\pi], \tag{18}$$

$$\begin{array}{rcl} v_1 & = & v\cos\theta_1, \\ v_2 & = & v\sin\theta_1\cos\theta_2, \\ \cdots & . & \cdots \\ v_{n-1} & = & v\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ v_n & = & v\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \end{array}$$

$$\mathrm{d}S_1 = \sin^{n-2}\theta_1 \sin^{n-1}\theta_2 \cdots \sin\theta_{n-2}\,\mathrm{d}\theta_1\mathrm{d}\theta_2\cdots\mathrm{d}\theta_{n-1}.$$

Thus for the exact solution of the problem (3) we obtain from the Eq.(12)

$$P(\lambda,\mathbf{v}) = \frac{\mathrm{e}^{\beta_0(\lambda)}}{\sqrt{4\pi\gamma(\lambda)}} \mathrm{e}^{\gamma(\lambda)\hat{\Lambda}_\theta} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4\gamma(\lambda)}} \varphi\left(\mathbf{v}\mathrm{e}^{\beta_1(\lambda)+(n-2)\gamma(\lambda)\pm s}\right)\mathrm{d}s \tag{19}$$

$$= \frac{\mathrm{e}^{\beta_0(\lambda)}}{\sqrt{4\pi\gamma(\lambda)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4\gamma(\lambda)}} \left[\sum_{l=0}^{\infty}\sum_{k=1}^{d_{l,n}} \mathrm{e}^{-\gamma(\lambda)\lambda_l}\, Y^{(k)}_{l,n}(\theta)\, \varphi_{l,k}\left(\mathbf{v}\mathrm{e}^{\beta_1(\lambda)+(n-2)\gamma(\lambda)\pm s}\right)\right]\mathrm{d}s,$$

where $\mathbf{v} \in \mathcal{R}^n$, $\beta_0(\lambda), \beta_1(\lambda)$ and $\gamma(\lambda)$ are from (9), and λ_l and $d_{l,n}$ are from (16).

In partucular, for $\mathbf{v} \in \mathcal{R}^3$, we have ($\theta_1 \equiv \theta$ and $\theta_2 \equiv \varphi$)

$$\left(\hat{\Lambda}_{\theta\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

$$P(\lambda,\mathbf{v}) = \frac{\mathrm{e}^{\beta_0(\lambda)}}{\sqrt{4\pi\gamma(\lambda)}} \mathrm{e}^{\gamma(\lambda)\hat{\Lambda}_\theta} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4\gamma(\lambda)}} \varphi\left(\mathbf{v}\mathrm{e}^{\beta_1(\lambda)+\gamma(\lambda)\pm s}\right)\mathrm{d}s \tag{20}$$

$$= \frac{\mathrm{e}^{\beta_0(\lambda)}}{\sqrt{4\pi\gamma(\lambda)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4\gamma(\lambda)}} \left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathrm{e}^{-\gamma(\lambda)l(l+1)}\, Y_l^m(\theta,\varphi)\, \varphi_{l,m}\left(\mathbf{v}\mathrm{e}^{\beta_1(\lambda)+\gamma(\lambda)\pm s}\right)\right]\mathrm{d}s,$$

where $Y_l^m(\theta,\varphi)$ are the Laplace functions on the 1-sphere in $\mathcal{R}^3$.

Substituting the expression (19) in the Eq. (3) one can see immediately that $P(\lambda,\mathbf{v})$ is a solution of the problem (3) and, according to the Cauchy theorem, it is the only classical solution of this problem.

3 Exact solution of the problem (5)

Proceeding as in Section 2, we have for the solution of the problem (5):

$$P(\lambda,\mathbf{v}) = \mathrm{e}^{\beta_0(\lambda)}\mathrm{e}^{\beta_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}+\gamma(\lambda)(\mathbf{v}\cdot\nabla_{\mathbf{v}})^2}\varphi(\mathbf{v}), \tag{21}$$

where $\beta_0(\lambda), \beta_2(\lambda)$ and $\gamma(\lambda)$ are from (9).

Since

$$\Big[\ \beta_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}\ ,\ \gamma(\lambda)(\mathbf{v}\cdot\nabla_{\mathbf{v}})^2\ \Big] = 0, \tag{22}$$

we can factorize in the Eq. (21):

$$P(\lambda,\mathbf{v}) = \mathrm{e}^{\beta_0(\lambda)}\mathrm{e}^{\beta_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}}\mathrm{e}^{\gamma(\lambda)(\mathbf{v}\cdot\nabla_{\mathbf{v}})^2}\varphi(\mathbf{v}) = \mathrm{e}^{\beta_0(\lambda)}\mathrm{e}^{\gamma(\lambda)(\mathbf{v}\cdot\nabla_{\mathbf{v}})^2}\mathrm{e}^{\beta_1(\lambda)\mathbf{v}\cdot\nabla_{\mathbf{v}}}\varphi(\mathbf{v}). \tag{23}$$

Now using (13) and the formula

$$\mathrm{e}^{\gamma(\lambda)(\mathbf{v}\cdot\nabla_{\mathbf{v}})^2}\varphi(\mathbf{v}) = \frac{1}{\sqrt{4\pi\gamma(\lambda)}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{s^2}{4\gamma(\lambda)}}\varphi\left(\mathbf{v}\mathrm{e}^{\pm s}\right)\mathrm{d}s, \tag{24}$$

we find for the exact solution of the problem (5):

$$P(\lambda,\mathbf{v}) = \frac{\mathrm{e}^{\beta_0(\lambda)}}{\sqrt{4\pi\gamma(\lambda)}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{s^2}{4\gamma(\lambda)}}\varphi\left(\mathbf{v}\mathrm{e}^{\beta_1(\lambda)\pm s}\right)\mathrm{d}s, \tag{25}$$

where $\beta_0(\lambda), \beta_1(\lambda)$ and $\gamma(\lambda)$ are defined in (9).

Substituting the expression (25) in the Eq.(5) one can check that $P(\lambda,\mathbf{v})$ is solution of the problem (5), and, according to the Cauchy theorem, it is the only classical solution of the problem.

4 Conclusions

The exact solutions of the Cauchy problems (3) and (5) are obtained, using the disentangling techniques of R. Feynman and M. Suzuki's methods for solving the 1-dimensional Fokker-Planck equation. The problems (3) and (5) are two of many n-dimensional generalizations of the 1-dimensional FPE, used by Friedrich and Peinke in their description of a turbulent cascade as a stochastic process of marcovian type.

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